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The Finite-Size Scaling Functions of the Four-Dimensional Ising Model

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A finite-size scaling function of the Privman–Fisher form is proposed for the singular part of the free-energy density of the four-dimensional Ising model. It leads to the finite-size scaling relations available and to the prediction of new ones.

KEY WORDS: Ising model; finite-size scaling.

The conventional finite-size scaling theory[1] is not applicable for the Ising model at and above the upper critical dimension $d_u = 4$. For the Ising model in the dimensionality $d > d_u$ the Privman–Fisher hypothesis[2] for the singular part of the free-energy density of a hypercubic finite system L^d of linear dimension L with periodic boundary conditions was adapted.[3] The predictions derived from it were tested and verified numerically for $d = 5$,[4] $d = 6$[5] and $d = 7$.[6] It was adapted also for the $O(N)$ model ($N \geqslant 2$) for a finite system having the general geometry $L^{(d-d')} \times \infty^{d'}$ ($d' \leqslant 2$) with periodic boundary conditions.[7] It reduces to the Ising case for $d' = 0$. For the four-dimensional Ising model the finite-size scaling relations are derived from the theories based on the renormalization group theory. They yield the free energy correct to leading logarithms. One of these,[8] called the renormalized mean field theory, gives a rounded peak for a finite-system phase transition. It predicts the finite-size scaling relations for the specific heat,[8,9] the magnetic susceptibility and the Binder cumulant.[9,10] In the other theory[11] a perturbative renormalization group method is used in deriving the free-energy density. By using the partition-function zeroes calculated from it the finite-size scaling relations for the specific heat and the

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magnetic susceptibility are obtained. Both theories predict $(T_c - T_c(L)) \propto L^{-2}\log^{-1/6} L$ for the finite-size shift of the critical temperature where $T_c(L)$ and T_c are the critical temperatures for the finite and infinite systems, respectively. For the $O(N)$ model in the large N limit the finite-size scaling relation for the magnetic susceptibility[12] is found to be the same as the one for the Ising model. The Privman–Fisher hypothesis[2] for the singular part of the free-energy density of a finite system having the general geometry $L^{(d-d')} \times \infty^{d'}$ $(d' \leqslant 2)$ with periodic boundary conditions was adapted also for the spherical model in $d = d_u$.[13]

The singular part of the free-energy density $f_L^{(S)}(t, h)$ of a hypercubic finite system L^d with periodic boundary conditions for $d < d_u$ is given by Privman and Fisher[2] as:

$$f_L^{(S)}(t, h) = L^{-d} Y(C_1 t L^{1/\nu}, C_2 h L^{\Delta/\nu}), \qquad t \to 0, \quad h \to 0, \quad L \to \infty \tag{1}$$

where Δ is the gap exponent, ν is the critical exponent for the correlation length for the infinite system, $t = (T - T_c)/T_c$ is the reduced temperature and h is the reduced external magnetic field. The scale factors C_1 and C_2 are the only nonuniversal system-dependent parameters, that is, the scaling function $Y(x, y)$ is universal, with no further nonuniversal prefactor.

In the present study the Privman–Fisher hypothesis for the singular part of the free-energy density $f_L^{(S)}(t, h)$ of a hypercubic finite system L^d with periodic boundary conditions is adapted for the Ising model in $d = 4$ dimensions, by proposing the finite-size scaling function $Y(x, y)$, correct to leading logarithms, as below:

$$f_L^{(S)}(t, h) = L^{-4} Y(C_1 t L^2 \log^{1/6} L, C_2 h L^3 \log^{1/4} L), \qquad t \to 0, \quad h \to 0, \quad L \to \infty \tag{2}$$

In getting this expression for $Y(x, y)$, $f_L^{(S)}(t, h)$ is assumed to have the Privman–Fisher form with the explicit L-dependence of x and y not known a priori. The expressions for the magnetic susceptibility $\chi_L(t, h)$ and the singular part of the specific heat $C_L^{(S)}(t, h)$ derived from it are evaluated at $t = 0$ and $h = 0$. These and the knowledge of $\chi_L(0, 0) \propto L^2 \log^{1/2} L$ and $C_L^{(S)}(0, 0) \propto \log^{1/3} L$[8, 9, 11] determine the L-dependence of x and y as in Eq. (2). For the Ising model in $d = d_u$, the expression for $f_L^{(S)}(t, h)$ derived starting with the renormalization group equations in differential form[14] reduces to the one given in Eq. (2) for $L \to \infty$ and confirms it. For the spherical model in $d = 4$ dimensions for a hypercube $(d' = 0)$ with periodic boundary conditions,[13, 15] $f_L^{(S)}(t, h)$ differs from the one in Eq. (2) for the Ising case in the first variable: $x = C_1 t L^2 \log^{-1/2} L$. From Eq. (2) the finite-size scaling expressions for the magnetization $M_L(t, h)$, the magnetic susceptibility $\chi_L(t,h)$, the singular part of the specific heat $C_L^{(S)}(t,h)$, and the Binder cumulant[10] $g_L(t,h)$ can be derived as below:

$$M_L(t,h) = -\frac{\partial f_L}{\partial h} = L^{-1}\log^{1/4}(L)\, C_2 U(C_1 t L^2 \log^{1/6} L, C_2 h L^3 \log^{1/4} L) \quad (3)$$

$$\chi_L(t,h) = -\frac{\partial^2 f_L}{\partial h^2} = L^2 \log^{1/2}(L)\, C_2^2 V(C_1 t L^2 \log^{1/6} L, C_2 h L^3 \log^{1/4} L) \quad (4)$$

$$C_L^{(S)}(t,h) = -\frac{\partial^2 f_L}{\partial t^2} = \log^{1/3}(L)\, C_1^2 W(C_1 t L^2 \log^{1/6} L, C_2 h L^3 \log^{1/4} L) \quad (5)$$

$$g_L(t,h) = \frac{\chi_L^{(4)}}{L^4 \chi_L^2} = G(C_1 t L^2 \log^{1/6} L, C_2 h L^3 \log^{1/4} L) \quad (6)$$

with the fourth derivative given by $\chi_L^{(4)} = -\partial^4 f_L/\partial h^4$. They can be rewritten in more informative forms as follows:

$$M_L(t,h) = L^{-\beta/\nu}\log^{1/4}(L)\, C_2 U(C_1 t L^2 \log^{1/6} L, C_2 h L^3 \log^{1/4} L) \quad (7)$$

$$\chi_L(t,h) = L^{\gamma/\nu}\log^{1/2}(L)\, C_2^2 V(C_1 t L^2 \log^{1/6} L, C_2 h L^3 \log^{1/4} L) \quad (8)$$

$$C_L^{(S)}(t,h) = L^{\alpha/\nu}\log^{1/3}(L)\, C_1^2 W(C_1 t L^2 \log^{1/6} L, C_2 h L^3 \log^{1/4} L),\ \alpha = 0 \quad (9)$$

where α, β, γ and ν are the critical exponents for the specific heat, the magnetization, the magnetic susceptibility and the correlation length of the infinite lattice, respectively; U, V, W and G are the corresponding finite-size scaling functions. For $h = 0$ they reduce to the following equations:

$$|M_L(t)| = L^{-\beta/\nu}\log^{1/4}(L)\, C_2 U(C_1 t L^2 \log^{1/6} L) \quad (10)$$

$U(C_1 t L^2 \log^{1/6} L)$ is identically zero for $M_L(t)$ because in the absence of symmetry-breaking fields the average of the magnetization is identically zero, but it is not for the absolute value of the magnetization $|M_L(t)|$.

$$\chi_L(t) = L^{\gamma/\nu}\log^{1/2}(L)\, C_2^2 V(C_1 t L^2 \log^{1/6} L) \quad (11)$$

$$C_L^{(S)}(t) = L^{\alpha/\nu}\log^{1/3}(L)\, C_1^2 W(C_1 t L^2 \log^{1/6} L), \qquad \alpha = 0 \quad (12)$$

$$g_L(t) = G(C_1 t L^2 \log^{1/6} L) \quad (13)$$

These predictions (Eqs. (11)–(13)) are the same as the ones given in refs. 8 and 9. For $h = 0$ and $t = 0$ they reduce to the following equations:

$$|M_L| = L^{-\beta/\nu}\log^{1/4}(L)\, C_2 U(0,0) \quad (14)$$

$$\chi_L = L^{\gamma/\nu}\log^{1/2}(L)\, C_2^2 V(0,0) \quad (15)$$

$$C_L^{(S)} = L^{\alpha/\nu} \log^{1/3}(L)\, C_1^2 W(0,0), \qquad \alpha = 0 \tag{16}$$

$$g_L = G(0,0) \tag{17}$$

$U(0,0)$, $V(0,0)$, $W(0,0)$ and $G(0,0)$ are universal. The relations with nonzero h (Eqs. (3)–(6)) and for $|M_L|$ are new. The relations for $h=0$ can be tested by simulations directly. For this purpose Monte Carlo simulations with Metropolis algorithm[16] are carried out on simple hypercubic lattices L^4 of linear dimensions $4 \leqslant L \leqslant 16$ with periodic boundary conditions. At T_c fifteen independent simulations are carried out, each one lasting 3×10^4 sweeps (6×10^4 sweeps for $L=16$, since $T_c(16)$ is nearest to T_c and the critical slowing down becomes more pronounced), for finding the mean values and the statistical errors. In computing the data for the finite-size scaling plots three runs are carried out at each temperature within the interval $5.98 \leqslant T \leqslant 7.38$ for the lattices $4 \leqslant L \leqslant 14$. The finite-size scaling plots for $|M_L(t)|$, $\chi_L(t)$, $C_L^{(S)}(t)$ and $g_L(t)$ are given in Figs. 1–4, respectively. The specific heat $C_L(t)$ contains, in addition to the singular part $C_L^{(S)}(t)$, a nonsingular part given by a constant b. $C_L(t)$ is obtained directly from the simulations and b is obtained as the value which makes the scaled $C_L^{(S)}(t)$ overlap best. The scaled data for different L overlap, as in

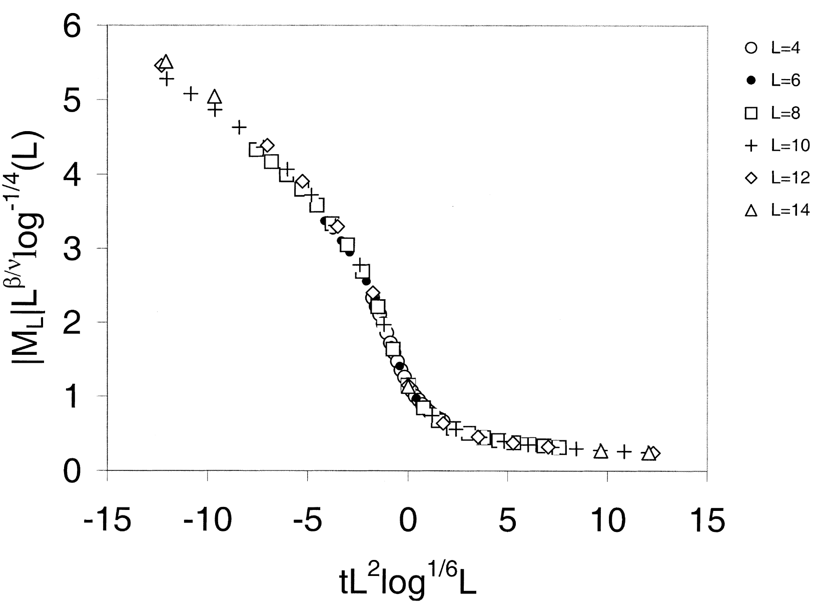

Fig. 1. The finite-size scaling plot of $|M_L|$ with $\beta/\nu = 1$ and $T_c = 6.6802$. The error bars are smaller than the symbols.

ref. 9, verifying the finite-size scaling relations given in Eqs. (10)–(13). As T goes away from T_c the points which do not satisfy the conditions of validity (Eq. (2)) for the finite-size scaling relations start to deviate from the curve formed by the overlapping parts of the plots for different L. The slopes of the straight lines fitting the log–log plots of $|M_L| \log^{-1/4}(L)$, $\chi_L \log^{-1/2}(L)$ and $C_L^{(S)} \log^{-1/3}(L)$ at $h=0$ and $t=0$ yield the values of the critical exponents β/ν, γ/ν (Fig. 5) and α/ν (Fig. 6), respectively. The Monte Carlo simulations of comparable quality[9] give powers of $\log L$ in agreement with the theoretical ones for χ_L, $C_L^{(S)}$ and $\chi_L^{(4)}$. The results are affected by the value of T_c. The values of T_c obtained by different methods are as follows: $T_c = 6.6817(15)$[17] (series expansion), 6.6802(2)[18] (series expansion), 6.6803(1)[18] (dynamic Monte Carlo), 6.680(1)[11] (cluster Monte Carlo), 6.680[19] (Creutz cellular automaton). $T_c = 6.6817(15)$ and 6.680(1) are used in ref. 9 and in refs. 19 and 20, respectively. In the present study $T_c = 6.6802(2)$[18] is used in finding the critical exponents and in plotting the finite-size scaling functions. The values of the critical exponents computed are as follows: $\alpha/\nu = 0.01(10)$ $(10 \leqslant L \leqslant 16)$, 0.04(6) $(8 \leqslant L \leqslant 16)$ and 0.04(5) $(6 \leqslant L \leqslant 16)$ for $b=0$, $\alpha/\nu = -0.01(5)$ $(8 \leqslant L \leqslant 16)$, $-0.01(4)$ $(6 \leqslant L \leqslant 16)$ and 0.01(3) $(4 \leqslant L \leqslant 16)$ for $b = -0.40(5)$ which makes the

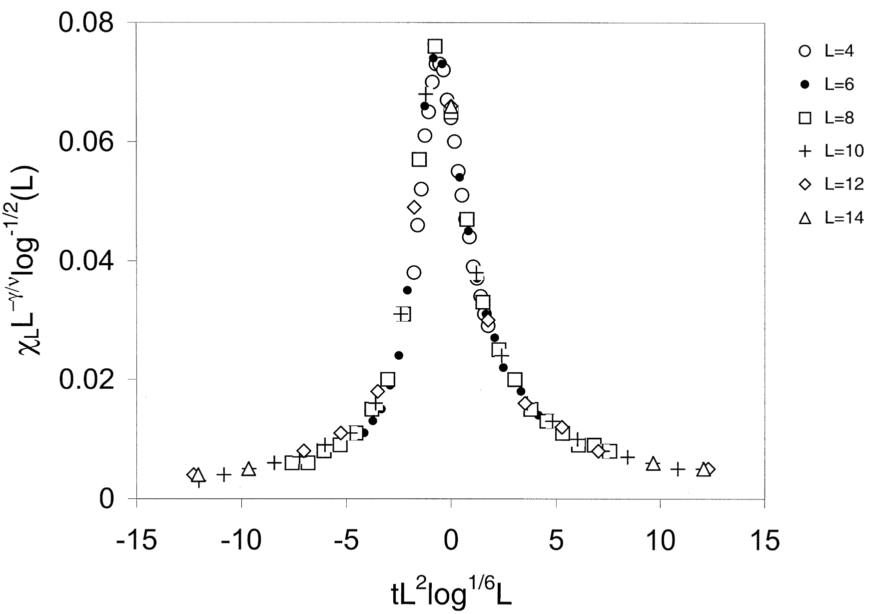

Fig. 2. The finite-size scaling plot of χ_L with $\gamma/\nu = 2$ and $T_c = 6.6802$. The error bars are smaller than the symbols or have about the same size.

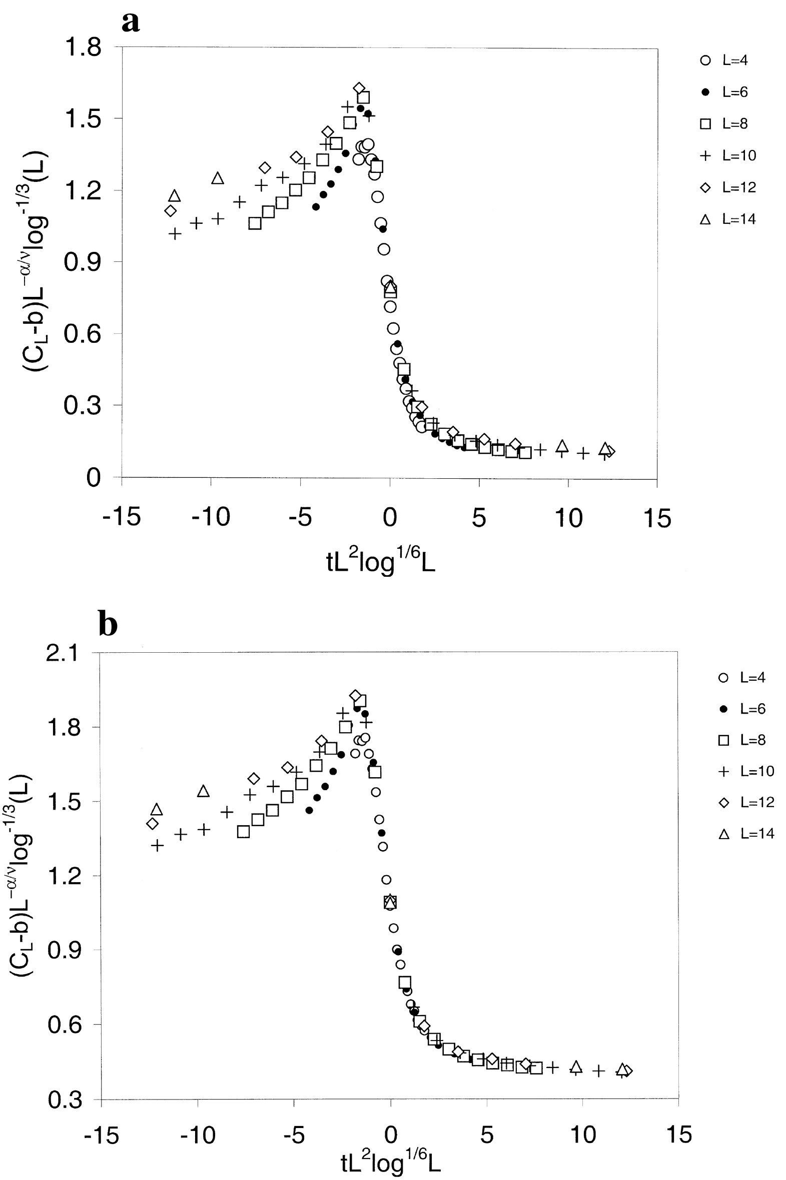

Fig. 3. (a) The finite-size scaling plot of the singular part of the specific heat $C_L^{(S)} = (C_L - b)$ with $\alpha/\nu = 0$, $T_c = 6.6802$ and the nonsingular part of the specific heat $b = 0$. The error bars are smaller than the symbols or have about the same size; (b) The finite-size scaling plot of the singular part of the specific heat $C_L^{(S)} = (C_L - b)$ with $\alpha/\nu = 0$, $T_c = 6.6802$ and the nonsingular part of the specific heat $b = -0.40(5)$ which makes the plots of the scaled $C_L^{(S)}$ overlap best.

scaled $C_L^{(S)}(t)$ overlap best (Fig. 3), $\beta/\nu = 1.02(2)$ ($4 \leqslant L \leqslant 16$) and $\gamma/\nu = 2.01(3)$ ($6 \leqslant L \leqslant 16$). The values of α/ν for $b = 0$ and Fig. 6 reveal that the effect of b on the value of α/ν is negligible at T_c for $L \geqslant 10$. The results of other studies using the simulations on the Creutz cellular automaton are as below ($b = 0$): $\alpha/\nu = -0.03$ ($6 \leqslant L \leqslant 14$)[20] and $\gamma/\nu = 1.97$ ($6 \leqslant L \leqslant 14$)[20] (each data point is the result of one run which lasts 3×10^4 sweeps). $\alpha/\nu = -0.036$ ($4 \leqslant L \leqslant 16$),[19] 0.006 ($8 \leqslant L \leqslant 16$),[19] -0.002 ($10 \leqslant L \leqslant 16$),[21] $\beta/\nu = 1.002$ ($4 \leqslant L \leqslant 16$)[21] and $\gamma/\nu = 2.003$ ($4 \leqslant L \leqslant 16$)[19] (each data point is the average of three runs each of which lasts 9.6×10^5 sweeps for $L \leqslant 10$, 3.6×10^5 sweeps for $L > 10$). These values of α/ν are in accordance with the above conclusion about the effect of b on the value of α/ν, and the values obtained in the present study for β/ν, γ/ν and α/ν ($10 \leqslant L \leqslant 16$, $b = 0$) are in good agreement with them. The present results are also in good agreement with the theoretical values $\alpha/\nu = 0$, $\beta/\nu = 1$ and $\gamma/\nu = 2$. For the Binder cumulant $(3 + g_L(0)) = 1/Q_L(0) = 1.92(3)$ is computed for $L = 14$, the same value as in ref. 9. Its theoretical value for the infinite lattice is $(3 + G(0)) = 1/Q(0) = 2.1884\ldots$.[22] The least-squares best fit of the data for $Q_L(0)$ to the expression $Q_L(0) = Q(0) + p/(L^2) + q/(\log L) + \ldots\ldots$[14, 23] with $Q(0)$, p and q being the fitting parameters, results

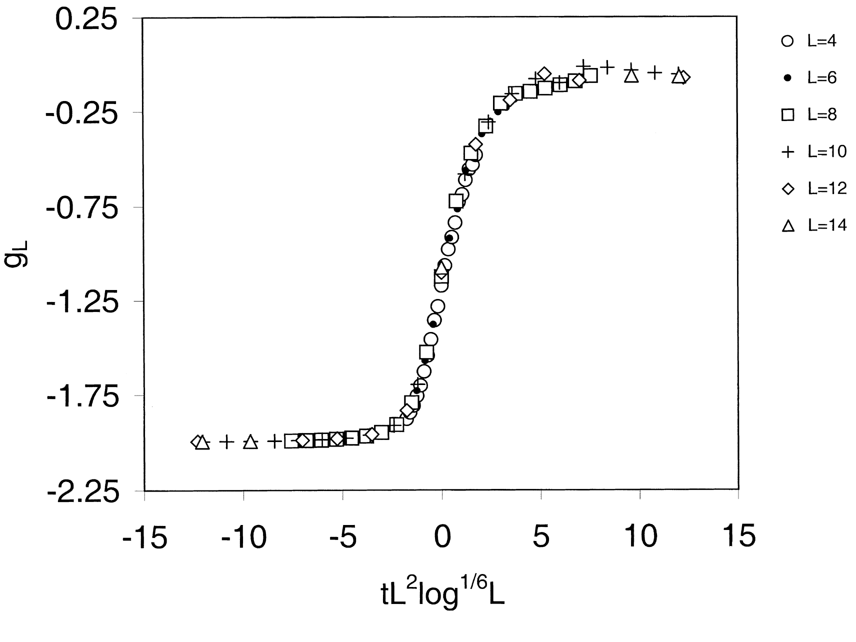

Fig. 4. The finite-size scaling plot of g_L with $T_c = 6.6802$. The error bars are smaller than the symbols or have about the same size.

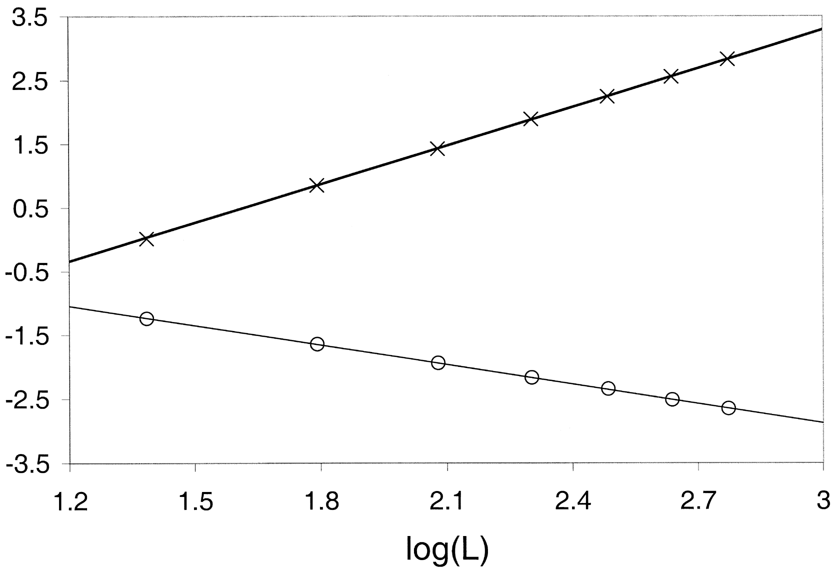

Fig. 5. The log-log plot of $|M_L| \log^{-1/4}(L)$ against L within the interval $4 \leqslant L \leqslant 16$ ($\circ$), and that of $\chi_L \log^{-1/2}(L)$ within the interval $6 \leqslant L \leqslant 16$ at $T_c = 6.6802$ ($\times$). The slopes give $\beta/\nu = 1.02(2)$ and $\gamma/\nu = 2.01(3)$. The error bars are smaller than the symbols.

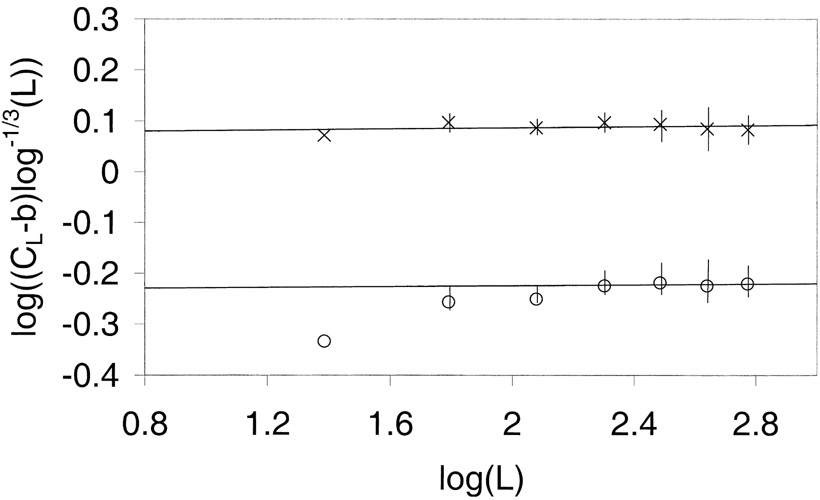

Fig. 6. The log-log plot of $(C_L - b) \log^{-1/3}(L)$ against L at $T_c = 6.6802$. For L within the interval $10 \leqslant L \leqslant 16$ and $b = 0$ ($\circ$), the slope gives $\alpha/\nu = 0.01(10)$; for L within the interval $4 \leqslant L \leqslant 16$ and $b = -0.40(5)$ ($\times$), the slope gives $\alpha/\nu = 0.01(3)$.

in $(3+G(0)) = 2.2(1)$ $(4 \leqslant L \leqslant 16)$ and 2.2(2) $(6 \leqslant L \leqslant 16)$. This value is in agreement with the theoretical one and with the results of other studies: 2.17(2) obtained from the simulation of the lattice gas model by the geometrical cluster Monte Carlo method,[23] and 2.16(2), 2.24(4) and 2.23(4) obtained from the simulation of the long-range Ising models in $d = 1$, 2 and 3 dimensions, respectively, by the cluster Monte Carlo method.[14]

The computer used is a Pentium-S with CPU at 166 Mhz. The CPU time invested is 1010 hours for the simulations at $T = T_c$, and 1592 hours for all the simulations. The corresponding values are 14 hours and 367 hours[24] in ref. 20, and 1090 hours and 10690 hours[21] in ref. 19.

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